# Benchmark Model with Intensity Based Jumps

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**Abstract.** This paper proposes a class of financial market models with security price processes that exhibit intensity based jumps. Primary security account prices, when expressed in units of the benchmark, turn out to be local martingales. The benchmark is chosen to be the growth optimal portfolio. The proposed benchmark model excludes, so called, benchmark arbitrage but permits arbitrage amounts, which arise for benchmarked price processes that are strict local martingales. In the proposed framework, generally, an equivalent risk neutral measure does not exist. Benchmarked fair derivative prices are obtained as conditional expectations of future benchmarked prices under the real world probability measure.

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# 1 Introduction

A rich literature has now emerged on continuous time asset price modeling and arbitrage pricing. For a recent account, see, for instance, Karatzas & Shreve (1998). In the standard risk neutral approach a major problem arises in modeling credit and insurance risk due to the difficulty in choosing an appropriate equivalent risk neutral pricing measure. On the other hand, the actuarial sciences have focused over the decades on the modeling and pricing of insurance risk under the real world probability measure with its seemingly separate set of concepts and methodologies, see, for instance, Gerber (1990) and Bühlmann (1992). New challenges are therefore arising from the need to have an integrated approach to the modeling of risk in the combined fields of finance and insurance.

This paper proposes an integrated approach that can be used for both finance and insurance applications. It uses the *growth optimal portfolio* (GOP) as benchmark and establishes a class of *benchmark models* with intensity based jumps. In the case of diffusions without jumps, Long (1990) and Bajeux-Besnainou & Portait (1997) have introduced the GOP as *numeraire portfolio*. Geman, El Karoui & Rochet (1995) developed an important *change of numeraire technique*. Kramkov & Schachermayer (1999), Becherer (2001), Goll & Kallsen (2002) and Platen (2002) derived conditions for semimartingale markets that guarantee the existence of a GOP, where benchmarked prices appear to be supermartingales.

Using the, so called, benchmark approach, see Platen (2002), we construct a class of benchmark models for security prices that follow diffusions with intensity based jumps. By avoiding the standard assumption on the existence of an equivalent risk neutral measure, freedom is gained for financial modeling. Benchmarked fair price processes are defined as conditional expectations of future benchmarked prices under the real world probability measure and are therefore martingales. If an equivalent risk neutral pricing measure exists, then fair and risk neutral prices coincide. For a wide range of applications in insurance the actuarial pricing formula can be explained as fair pricing.

The benchmark approach is in line with the view expressed in Bühlmann, Delbaen, Embrechts & Shiryaev (1998), where in a discrete time setting the importance of the state price deflator in both financial and insurance modeling is emphasized, see also Duffie (1996), Rogers (1997) and Bühlmann & Platen (2002). Classical results on no-arbitrage pricing that assume an equivalent pricing measure, see Ross (1976), Harrison & Kreps (1979), Harrison & Pliska (1981), Föllmer & Sondermann (1986), Föllmer & Schweizer (1991) and Delbaen & Schachermayer (1998), follow naturally under the benchmark approach.

A natural form of the risk premia in the presence of intensity based jumps is identified. It captures that of the *intertemporal capital asset pricing model*, see Merton (1973), if one interprets the GOP as the market portfolio. The proposed benchmark model will be shown to exclude, so called, benchmark arbitrage. However, it still permits certain arbitrage amounts. Strictly positive arbitrage amounts arise when benchmarked prices are strict local martingales. The benchmark model provides an alternative and rich modeling framework, where all major quantitative tasks, including calibration, derivative pricing, hedging and portfolio optimization are conducted in an integrated manner under the real world probability measure.

The paper introduces in Section 2 a continuous time benchmark model with intensity based jumps. Benchmark arbitrage, arbitrage amounts and fair pricing are studied in Section 3. Section 4 describes a multi-factor benchmark model and discusses an example.

# 2 Market with Intensity Based Jumps

## 2.1 Uncertainty

Let us model continuously evolving uncertainty by $m$ independent standard Wiener processes $W^k = \{W^k(t),\, t \in [0,T]\}$, $k \in \{1,2,\dots,m\}$, $m \in \{1,2,\dots,d\}$, defined on a filtered probability space $(\Omega, \mathcal{A}_T, \underline{\mathcal{A}}, P)$ with finite time horizon $T \in (0,\infty)$. We model events of certain types, for instance, the default of a particular company, operational failures or specified insured events. Events of the $k$th type are counted by the $\underline{\mathcal{A}}$-adapted, independent $k$th *counting process* $p^k = \{p^k(t),\, t \in [0,T]\}$, where its *intensity* $h^k = \{h^k_t,\, t \in [0,T]\}$ is a given, predictable, strictly positive process, $k \in \{m+1,\dots,d\}$. Furthermore, we introduce the $k$th *jump martingale* $W^k = \{W^k(t),\, t \in [0,T]\}$ with stochastic differential

$$dW^k(t) = dp^k(t) - h^k_t\, dt \tag{2.1}$$

for $k \in \{m+1,\dots,d\}$ and $t \in [0,T]$. The above jump martingales are the compensated building blocks for the modeling of event driven uncertainty.

The total evolution of uncertainty is modeled by the vector of independent $(\underline{\mathcal{A}}, P)$-martingales $W(t) = \{(W^1(t),\dots,W^d(t))^\top,\, t \in [0,T]\}$. Here we denote by $A^\top$ the transpose of a vector or matrix $A$. Note, we aggregate in $W$ the Wiener processes and the jump martingales. The filtration $\underline{\mathcal{A}} = (\mathcal{A}_t)_{t\in[0,T]}$ is the augmentation under $P$ of the natural filtration $\mathcal{A}^W$, generated by $W$, fulfilling the usual conditions, where $\mathcal{A}_0$ is the trivial $\sigma$-algebra, see Protter (1990). The increments $W^k(t+\varepsilon) - W^k(t)$ are assumed to be independent of $\mathcal{A}_t$ for $t \in [0,T]$ and any $\varepsilon > 0$, $k \in \{1,2,\dots,d\}$, see Protter (1990).

## 2.2 Primary Security Accounts and Strategies

Let us introduce $d+1$ *primary security accounts*, $d \in \{1,2,\dots\}$. These accounts hold the corresponding primary security as well as the accrued income or loss generated by the corresponding primary asset, for instance, a currency, commodity or share. The 0th primary security account $S^{(0)} = \{S^{(0)}(t),\, t \in [0,T]\}$ is the *domestic savings account process*, where

$$dS^{(0)}(t) = S^{(0)}(t)\, r(t)\, dt \tag{2.2}$$

for $t \in [0,T]$, with initial value $S^{(0)}(0) = 1$. This savings account is assumed to continuously accrue the interest $r(t)$ from holding the domestic currency.

The evolution of the value of the $j$th primary security, $j \in \{1,2,\dots,d\}$, is modeled by the $j$th *primary security account process* $S^{(j)} = \{S^{(j)}(t),\, t \in [0,T]\}$. For a foreign currency this is the corresponding foreign savings account value, when expressed in units of the domestic currency. In the case of shares, $S^{(j)}$ is the share price process with all dividends reinvested. The vector process $S = \{S(t) = (S^{(0)}(t),\dots,S^{(d)}(t))^\top,\, t \in [0,T]\}$ characterizes the evolution of the primary security accounts. Assume that the nonnegative $j$th primary security account value $S^{(j)}(t)$ at time $t$ satisfies the stochastic differential equation (SDE)

$$\begin{aligned} dS^{(j)}(t) \;&=\; S^{(j)}(t-)\Bigg\{ r(t)\,dt + \sum_{k=1}^{m} (\sigma^{0,k}(t) - \sigma^{j,k}(t))\,(\sigma^{0,k}(t)\,dt + dW^k(t)) \\ &\quad + \sum_{k=m+1}^{d} \left( \frac{\varphi^{j,k}_{t-}}{\varphi^{0,k}_{t-}} - 1 \right) \left( dp^k(t) - \varphi^{0,k}_{t-}\, h^k_{t-}\, dt \right) \Bigg\} \end{aligned} \tag{2.3}$$

for $t \in [0,T]$ with initial value $S^{(j)}(0) > 0$, $j \in \{1,2,\dots,d\}$, see Protter (1990). To ensure that the primary security accounts exist and remain nonnegative we have to assume that

$$\varphi^{0,k}_t > 0 \tag{2.4}$$

and

$$\varphi^{j,k}_t \geq 0 \tag{2.5}$$

a.s. for $t \in [0,T]$, $j \in \{1,2,\dots,d\}$ and $k \in \{m+1,\dots,d\}$. Furthermore, we assume that

$$\begin{aligned} &\int_0^T \Bigg\{ r(s) + \sum_{j=0}^{d} \Bigg( \sum_{k=1}^{m} (\sigma^{j,k}(s))^2 \\ &+ \sum_{k=m+1}^{d} \left\{ \left( \frac{\varphi^{j,k}_s}{\varphi^{0,k}_s} - 1 \right)^2 + h^k_t + \left(\varphi^{0,k}_s - 1\right)^2 \right\} \Bigg) \Bigg\}\, ds < \infty \end{aligned} \tag{2.6}$$

a.s. and that a pathwise unique solution of (2.3) exists, see Protter (1990).

Let us set

$$b^{j,k}(t) = \begin{cases} \sigma^{0,k}(t) - \sigma^{j,k}(t) & \text{for} \quad k \in \{1, 2, \dots, m\} \\ \left(\frac{\varphi_t^{j,k}}{\varphi_t^{0,k}} - 1\right) & \text{for} \quad k \in \{m+1, \dots, d\} \end{cases} \tag{2.7}$$

for $t \in [0, T]$ and $j \in \{0, 1, \dots, d\}$. We call $b(t) = [b^{j,k}(t)]_{j,k=1}^d$ the *generalized volatility matrix*.

There is no loss of generality by parameterizing the coefficients as shown in (2.3). One can reparameterize the SDE (2.3) by using the generalized volatilities, see (2.7), and corresponding appreciation rates to obtain a more common form for the coefficients of the SDE (2.3).

We say that a predictable stochastic process $\delta = \{\delta(t) = (\delta^{(0)}(t), \dots, \delta^{(d)}(t))^\top, t \in [0, T]\}$ is a *strategy*, if $\delta$ is $S$-integrable, see Protter (1990). The $j$th component $\delta^{(j)}(t)$ of the strategy $\delta$ denotes the number of units of the $j$th primary security account, which are held at time $t \in [0, T]$ in the corresponding portfolio, $j \in \{0, 1, \dots, d\}$. For a strategy $\delta$ we denote by $S^{(\delta)}(t)$ the value of the corresponding portfolio at time $t$ when measured in units of the domestic currency, this means

$$S^{(\delta)}(t) = \delta(t)^\top S(t). \tag{2.8}$$

**Definition 2.1** *A strategy $\delta$ and the corresponding portfolio process $S^{(\delta)}$ are called* self-financing *if*

$$dS^{(\delta)}(t) = \delta(t-)^\top dS(t) \tag{2.9}$$

*for all $t \in [0, T]$.*

Under a self-financing strategy no outflow or inflow of funds occurs for the corresponding portfolio $S^{(\delta)}$ and all changes in the value of this portfolio are due to corresponding gains from trade. Since we will consider in the following only self-financing strategies and corresponding self-financing portfolios, we omit from now on the word "self-financing".

## 2.3 Benchmark Portfolio

For a given strategy $\delta$ with a.s. strictly positive portfolio value $S^{(\delta)}(t)$ at time $t$ let $\pi_\delta^j(t)$ denote the corresponding $j$th *proportion* of the value of this portfolio that is invested at time $t$ in the $j$th primary security account. This proportion is defined by the relation

$$\pi_\delta^j(t) = \delta^{(j)}(t) \, \frac{S^{(j)}(t)}{S^{(\delta)}(t)} \tag{2.10}$$

for $t \in [0,T]$ and $j \in \{0,1,\dots,d\}$. Furthermore, by (2.8) the proportions always add to one, that is

$$\sum_{j=0}^{d} \pi_\delta^j(t) = 1 \tag{2.11}$$

for $S^{(\delta)}(t) > 0$ a.s.

The vector of proportions $\pi_\delta(t) = (\pi_\delta^1(t), \dots, \pi_\delta^d(t))^\top$, see (2.10), allows us to obtain for $S^{(\delta)}(t)$ with (2.9), (2.3) and (2.10) the SDE

$$\begin{aligned} dS^{(\delta)}(t) = S^{(\delta)}(t-)\Bigg\{ r(t)\,dt + \sum_{k=1}^{m}\left(\sigma^{0,k}(t) - \sum_{j=0}^{d}\pi_\delta^j(t)\,\sigma^{j,k}(t)\right)(\sigma^{0,k}(t)\,dt + dW^k(t)) \\ + \sum_{k=m+1}^{d}\left(\frac{\sum_{j=0}^{d}\pi_\delta^j(t-)\,\varphi_{t-}^{j,k}}{\varphi_{t-}^{0,k}} - 1\right)\left(dp^k(t) - \varphi_{t-}^{0,k}\,h_{t-}^k\,dt\right)\Bigg\}, \end{aligned} \tag{2.12}$$

as long as $S^{(\delta)}(t) > 0$ a.s. for $t \in [0,T]$.

Let us introduce the vector $c_1(t) = (c_1^1(t), \dots, c_1^d(t))^\top$ with components

$$c_1^k(t) = \begin{cases} \sigma^{0,k}(t) & \text{for} \quad k \in \{1,2,\dots,m\} \\ \left(\frac{1}{\varphi_t^{0,k}} - 1\right) & \text{for} \quad k \in \{m+1,\dots,d\} \end{cases} \tag{2.13}$$

and $t \in [0,T]$. This allows us to define the *benchmark portfolio* $S^{(\underline{\delta})}$ by the corresponding vector of proportions

$$\pi_{\underline{\delta}}(t) = (\pi_{\underline{\delta}}^1(t), \dots, \pi_{\underline{\delta}}^d(t))^\top = \left(c_1(t)^\top\, b^{-1}(t)\right)^\top \tag{2.14}$$

for $t \in [0,T]$. It is straightforward to show that the value of the benchmark portfolio $S^{(\underline{\delta})}(t)$ at time $t$ satisfies the SDE

$$\begin{aligned} dS^{(\underline{\delta})}(t) &= S^{(\underline{\delta})}(t-)\Bigg\{ r(t)\,dt + \sum_{k=1}^{m}\sigma^{0,k}(t)\,(\sigma^{0,k}(t)\,dt + dW^k(t)) \\ &\quad + \sum_{k=m+1}^{d}\left(\frac{1}{\varphi_{t-}^{0,k}} - 1\right)\left(dp^k(t) - \varphi_{t-}^{0,k}\,h_{t-}^k\,dt\right)\Bigg\} \end{aligned} \tag{2.15}$$

for $t \in [0,T]$, where we set

$$S^{(\underline{\delta})}(0) = 1. \tag{2.16}$$

In the case without jumps, if we interpret the above benchmark portfolio as the market portfolio of the *intertemporal capital asset pricing model* (ICAPM), see

Merton (1973), then we can read off the corresponding well-known form of the risk premia from the SDE (2.12). The risk premium with respect to the $k$th Wiener process at time $t$ is then

$$p^{\delta,k}(t) = \left(\sigma^{0,k}(t) - \sum_{j=0}^{d} \pi_\delta^j(t)\, \sigma^{j,k}(t)\right) \sigma^{0,k}(t) \tag{2.17}$$

for $t \in [0,T]$ and $k \in \{1, 2, \ldots, m\}$. As in the ICAPM, (2.17) reflects the correlation between the returns of the portfolio and the benchmark portfolio with respect to the corresponding Wiener process. Additionally, we obtain from (2.12) the *jump risk premium* with respect to the $k$th jump martingale. It has the form

$$p^{\delta,k}(t) = \left(\frac{\sum_{j=0}^{d} \pi_\delta^j(t)\, \varphi^{j,k}(t)}{\varphi_t^{0,k}} - 1\right) \left(1 - \varphi_t^{0,k}\right) h_t^k \tag{2.18}$$

for $k \in \{m+1, \ldots, d\}$ and $t \in [0,T]$. Note, if there is no jump with respect to the $k$th jump martingale in the benchmark portfolio, that is $\varphi_t^{0,k} = 1$, then there is no contribution to the corresponding jump risk premium. Similarly, if there is no jump in the portfolio with respect to the $k$th jump martingale, that is $\frac{\sum_{j=0}^{d} \pi_\delta^j(t)\, \varphi^{j,k}(t)}{\varphi_t^{0,k}} = 1$, then there is also no corresponding jump risk premium. Only when the portfolio and the benchmark portfolio jump jointly, then a nonzero jump risk premium arises, which expresses the joint effect similar to the correlation between returns with respect to Wiener processes. If we consider the jump risk premium for the benchmark portfolio with respect to the $k$th jump martingale, then we get the interesting expression

$$p^{\underline{\delta},k}(t) = \left(\left(\varphi_t^{0,k}\right)^{-\frac{1}{2}} - \left(\varphi_t^{0,k}\right)^{\frac{1}{2}}\right)^2 \tag{2.19}$$

for $k \in \{m+1, \ldots, d\}$ and $t \in [0,T]$. Note, if the benchmark portfolio has no jumps, then there is no premium for any jump risk.

From now on we use the benchmark portfolio $S^{(\underline{\delta})}$ as reference unit and call prices when expressed in units of $S^{(\underline{\delta})}$ *benchmarked prices.*

## 2.4 Benchmark Model

Note that we did not impose any major restrictions on the dynamics of the primary security accounts. These are supposed to be nonnegative and have to satisfy the general SDE (2.3).

**Definition 2.2** *We call a model, which is expressed by the above collection* $(\Omega, \mathcal{A}_T, \underline{\mathcal{A}}, P, S)$, *a* benchmark model.

In the following we only consider benchmark models. The $j$th *benchmarked primary security account process* $\hat{S}^{(j)} = \{\hat{S}^{(j)}(t),\ t \in [0,T]\}$ with

$$\hat{S}^{(j)}(t) = \frac{S^{(j)}(t)}{S^{(\underline{\delta})}(t)} \tag{2.20}$$

for $t \in [0,T]$ and $j \in \{0,1,\dots,d\}$ satisfies by (2.3) and (2.15) the SDE

$$d\hat{S}^{(j)}(t) = \hat{S}^{(j)}(t-) \left[ -\sum_{k=1}^{m} \sigma^{j,k}(t)\, dW^k(t) + \sum_{k=m+1}^{d} (\varphi_{t-}^{j,k} - 1)\, (dp^k(t) - h_{t-}^k\, dt) \right] \tag{2.21}$$

for $t \in [0,T]$. Note that $\hat{S}^{(j)}$ in (2.21) is driftless and thus, see Protter (1990), an $(\underline{\mathcal{A}}, P)$-local martingale. This means, in general, $\hat{S}^{(j)}$ may be a strict local martingale, as we will see later on in Section 4.4. We observe from (2.21) that the volatilities $\sigma^{j,k}$ and jump ratios $\varphi^{j,k}$ provide an efficient parameterization of the SDE of the $j$th benchmarked primary security. This parameterization reflects a natural similarity of the SDEs of all benchmarked primary security accounts, including that of the benchmarked domestic savings account $\hat{S}^{(0)}$.

For any portfolio $S^{(\delta)}$ we introduce its *benchmarked portfolio value*

$$\hat{S}^{(\delta)}(t) = \frac{S^{(\delta)}(t)}{S^{(\underline{\delta})}(t)} \tag{2.22}$$

at time $t \in [0,T]$. By application of the Itô formula together with (2.12) and (2.15) we obtain the following statement.

**Corollary 2.3** *For any portfolio $S^{(\delta)}$ the corresponding benchmarked portfolio value $\hat{S}^{(\delta)}(t)$ satisfies the SDE*

$$\begin{aligned} d\hat{S}^{(\delta)}(t) &= -\sum_{k=1}^{m} \sum_{j=0}^{d} \delta^{(j)}(t)\, \hat{S}^{(j)}(t)\, \sigma^{j,k}(t)\, dW^k(t) \\ &\quad + \sum_{k=m+1}^{d} \left( \sum_{j=0}^{d} \delta^{(j)}(t-)\, \hat{S}^{(j)}(t-)\, \varphi_{t-}^{j,k} - \hat{S}^{(\delta)}(t-) \right) (dp^k(t) - h_{t-}^k\, dt) \end{aligned} \tag{2.23}$$

*for $t \in [0,T]$.*

The first term on the right hand side of (2.23) forms a continuous $(\underline{\mathcal{A}}, P)$-local martingale and the second term an $(\underline{\mathcal{A}}, P)$-martingale. Due to a result in Ansel & Stricker (1994), a nonnegative, local martingale is a supermartingale. This permits us to formulate the following important result.

**Corollary 2.4** *Any benchmarked, nonnegative portfolio process*

$$\hat{S}^{(\delta)} = \{\hat{S}^{(\delta)}(t) = \frac{S^{(\delta)}(t)}{S^{(\delta_*)}(t)},\ t \in [0,T]\}$$

*is an* $(\underline{\mathcal{A}}, P)$*-supermartingale.*

The supermartingale property obtained for benchmarked nonnegative portfolios, stems from the fact that the benchmark portfolio is in some sense the best performing portfolio, as we will see below.

## 2.5 Growth Optimal Portfolio

For a nonnegative portfolio process $S^{(\delta)}$ we define its *growth rate* $g_{t,s}^{(\delta)}$ over the period $[t, s]$ as the conditional expectation

$$g_{t,s}^{(\delta)} = E\left(\log\left(\frac{S^{(\delta)}(s)}{S^{(\delta)}(t)}\right) \Big| \mathcal{A}_t\right) \tag{2.24}$$

for $t \in [0,T]$ and $s \in [t,T]$, where we set $\log(0) = -\infty$.

**Definition 2.5** *If there exists a strictly positive portfolio* $S^{(\delta_*)}$ *such that for all* $t \in [0,T]$*,* $s \in [t,T]$ *and each nonnegative portfolio* $S^{(\delta)}$ *it holds that*

$$g_{t,s}^{(\delta)} \leq g_{t,s}^{(\delta_*)} < \infty, \tag{2.25}$$

*then we call* $S^{(\delta_*)}$ *a* growth optimal portfolio *(GOP).*

By application of Corollary 3.7 in Platen (2002) one obtains directly from Corollary 2.4 the following result.

**Corollary 2.6** *There exists a unique GOP* $S^{(\delta_*)}$*, which equals the benchmark portfolio* $S^{(\underline{\delta})}$*.*

By the supermartingale property of a benchmarked, nonnegative portfolio process $\hat{S}^{(\delta)}$, see Corollary 2.4, its expected future benchmarked value $E(\hat{S}^{(\delta)}(s)\,|\,\mathcal{A}_t)$ cannot be larger than its last observed benchmarked value $\hat{S}^{(\delta)}(t)$ for $0 \leq t \leq s \leq T$. Thus, in the sense of conditional expectations, the GOP can be interpreted as the *best benchmark portfolio.*

Obviously, the benchmarked value $\hat{S}^{(\delta_*)}(t) = \hat{S}^{(\underline{\delta})}(t) = 1$ of the GOP always remains equal to one. Consequently, substituting the strategy $\delta_* = \underline{\delta}$ of the GOP into (2.23) reveals particular properties of volatilities and jump ratios. With the proportions $\pi_{\delta_*}(t)$, given in (2.14), we obtain from (2.23) and (2.10) the following result.

**Corollary 2.7** *It holds*

$$\sum_{j=0}^{d} \pi^j_{\delta_*}(t)\, \sigma^{j,k}(t) = 0 \tag{2.26}$$

*for* $k \in \{1, 2, \ldots, m\}$ *and*

$$\sum_{j=0}^{d} \pi^j_{\delta_*}(t) \left(\varphi^{j,k}_t - 1\right) = 0 \tag{2.27}$$

*for* $k \in \{m+1, \ldots, d\}$ *and* $t \in [0, T]$.

Relation (2.26) shows that the average of the volatilities with respect to a given Wiener process is zero when these are weighted by the GOP proportions. Furthermore, it follows from (2.27) that the average of the jump ratios, with respect to a given jump martingale, equals one when weighted by the GOP proportions. These relations represent a conservation law for volatilities and jump ratios. Since volatilities and jump ratios measure risk this can be interpreted as *conservation of risk.*

The *optimal growth rate* $g_{\delta_*}(t)$ at time $t$, attained by the GOP $S^{(\delta_*)}$, has according to (2.15) and (2.24) and by application of the Itô formula the value

$$g_{\delta_*}(t) = \lim_{\varepsilon \to 0} g^{(\delta_*)}_{t,t+\varepsilon} = r(t) + \frac{1}{2} \sum_{k=1}^{m} (\sigma^{0,k}(t))^2 + \sum_{k=m+1}^{d} h^k_t \left(\varphi^{0,k}_t - 1 - \log(\varphi^{0,k}_t)\right) \tag{2.28}$$

for $t \in [0, T]$. We observe in the optimal growth rate (2.28) a third term additionally to the interest rate and the well-known second term, which equals half the GOP volatility, see Long (1990) or Karatzas & Shreve (1998). The third term maximizes the impact of the different types of events on the growth rate of the GOP. Note that this term is larger for larger intensities. All jump ratios, either strictly greater or smaller than one, contribute positively to the optimal growth rate. Their contribution is larger for jump ratios that are further away from one.

# 3 Benchmark Arbitrage and Fair Pricing

## 3.1 Benchmark Arbitrage

The literature describes various notions of arbitrage, see, for instance, Ross (1976), Harrison & Kreps (1979), Harrison & Pliska (1981), Föllmer & Sondermann (1986), Föllmer & Schweizer (1991), Delbaen & Schachermayer (1994, 1998), Karatzas & Shreve (1998) and Shiryaev & Cherny (2001). Most of these notions are tied to some equivalent risk neutral measure and an upper bound for the debt permitted. The bound for the debt is in the literature typically expressed in units of the domestic savings account. We use here a slightly different definition and introduce, so called, *benchmark arbitrage*. This notion avoids any reference to some arbitrary primary security account by using the GOP as reference unit. A natural upper bound of zero for the debt is chosen by simply focusing on nonnegative portfolios.

**Definition 3.1** *A benchmarked, nonnegative portfolio process* $\hat{S}^{(\delta)} = \{\hat{S}^{(\delta)}(t) = \frac{S^{(\delta)}(t)}{S^{(\delta_*)}(t)},\ t \in [0,T]\}$ *permits* benchmark arbitrage *if for some stopping times* $\tau \in [0,T]$ *and* $\sigma \in [0,\tau]$, *where*

$$\hat{S}^{(\delta)}(\tau) \geq \hat{S}^{(\delta)}(\sigma) \geq 0 \tag{3.1}$$

*a.s., it holds that*

$$P\left(\hat{S}^{(\delta)}(\tau) > \hat{S}^{(\delta)}(\sigma) \,\middle|\, \mathcal{A}_\sigma\right) > 0. \tag{3.2}$$

This means, in the case of benchmark arbitrage and zero initial capital $\hat{S}^{(\delta)}(\sigma) = 0$, it is possible to generate strictly positive wealth with strictly positive probability. In the case of strictly positive initial wealth $\hat{S}^{(\delta)}(\sigma) > 0$, the above definition expresses the fact that under benchmark arbitrage it is possible to systematically outperform the GOP with some strictly positive probability. One can now prove the following result.

**Corollary 3.2** *The given benchmark model does not permit benchmark arbitrage.*

**Proof**: Consider a benchmarked, nonnegative portfolio process $\hat{S}^{(\delta)}$, where, as in Definition (3.1), we have for stopping times $\sigma$ and $\tau$ the inequality

$$\hat{S}^{(\delta)}(\tau) \geq \hat{S}^{(\delta)}(\sigma) \tag{3.3}$$

with $0 \leq \sigma \leq \tau \leq T$ a.s. It follows by the supermartingale property of $\hat{S}^{(\delta)}$, see Corollary 2.4, and the Optional Sampling Theorem, see Protter (1990), that

$$E\left(\hat{S}^{(\delta)}(\tau) - \hat{S}^{(\delta)}(\sigma) \,|\, \mathcal{A}_\sigma\right) \leq 0. \tag{3.4}$$

The relations (3.3) and (3.4) induce that $\hat{S}^{(\delta)}(\tau)$ cannot be strictly greater than $\hat{S}^{(\delta)}(\sigma)$ with any strictly positive probability. Thus relation (3.2) does not hold, which proves that the given portfolio does not permit benchmark arbitrage. $\square$

## 3.2 Arbitrage Amounts

It is of interest to study the difference between the last observed benchmarked value of a portfolio and the conditional expectation of any of its future benchmarked values. For a nonnegative portfolio process $S^{(\delta)}$ we define the corresponding *arbitrage amount* $A_\delta(t)$ at time $t \in [0,T]$ as the difference

$$A_\delta(t) = S^{(\delta)}(t) - S^{(\delta_*)}(t)\, E\left(\hat{S}^{(\delta)}(T) \,|\, \mathcal{A}_t\right). \tag{3.5}$$

Due to the supermartingale property of $\hat{S}^{(\delta)}$, see Corollary 2.4, we obtain for the above form of arbitrage the following result.

**Corollary 3.3** *The arbitrage amount $A_\delta(t)$ is always* nonnegative*, that is*

$$A_\delta(t) \geq 0 \tag{3.6}$$

*a.s. for all $t \in [0,T]$.*

This means, a benchmark model permits strictly positive arbitrage amounts, as we will see in an example at the end of Section 4.4. This does not mean that there is any benchmark arbitrage, as described by Definition 3.1.

## 3.3 Risk Neutral Pricing

To highlight the relationship between the classical risk neutral and benchmark approaches, let us, for this subsection *only*, consider the case of a continuous market with equivalent risk neutral pricing measure $\tilde{P}$. According to the standard risk neutral pricing methodology for continuous markets, see Karatzas & Shreve (1998), any discounted portfolio process $\bar{S}^{(\delta)} = \{\bar{S}^{(\delta)}(t) = \frac{S^{(\delta)}(t)}{S^{(0)}(t)},\, t \in [0,T]\}$ forms then an $(\underline{\mathcal{A}}, \tilde{P})$-martingale. This means, denoting by $\tilde{E}$ expectation with respect to $\tilde{P}$, we have in this case the *risk neutral pricing formula*

$$S^{(\delta)}(t) = \tilde{E}\left(\frac{S^{(0)}(t)}{S^{(0)}(s)}\, S^{(\delta)}(s) \,\middle|\, \mathcal{A}_t\right) \tag{3.7}$$

for all $s \in [0,T]$ and $t \in [0,s]$. With Radon-Nikodym derivative process $\Lambda = \{\Lambda(t),\, t \in [0,T]\}$, where

$$\Lambda(t) = \frac{S^{(\delta_*)}(0)}{S^{(\delta_*)}(t)}\, \frac{S^{(0)}(t)}{S^{(0)}(0)} = \frac{\hat{S}^{(0)}(t)}{\hat{S}^{(0)}(0)} \tag{3.8}$$

and

$$\Lambda(T) = \frac{d\tilde{P}}{dP} \tag{3.9}$$

we can then write (3.7) in the form

$$S^{(\delta)}(t) = E\left(\frac{\Lambda(s)}{\Lambda(t)}\,\frac{S^{(0)}(t)}{S^{(0)}(s)}\,S^{(\delta)}(s)\,\bigg|\,\mathcal{A}_t\right). \tag{3.10}$$

Therefore, we obtain the pricing formula

$$S^{(\delta)}(t) = S^{(\delta_*)}(t)\,E\left(\hat{S}^{(\delta)}(s)\,|\,\mathcal{A}_t\right) \tag{3.11}$$

for $s \in [0, T]$ and $t \in [0, s]$. Note that the conditional expectation in (3.11) is taken with respect to the real world probability measure $P$. It involves the GOP $S^{(\delta_*)}$ but does not depend on the measure $\tilde{P}$. Furthermore, using (2.22), we can rewrite (3.11) in the benchmarked world as

$$\hat{S}^{(\delta)}(t) = E\left(\hat{S}^{(\delta)}(s)\,|\,\mathcal{A}_t\right) \tag{3.12}$$

for $s \in [0, T]$ and $t \in [0, s]$. This means, in the given case, where both (3.7) and (3.11) are assumed to be valid and $\tilde{P}$ is equivalent to $P$, risk neutral and benchmark pricing coincide if one chooses the formula (3.12) or equivalently (3.11). However, in general, this equivalence may break down if the benchmarked price process $\hat{S}^{(\delta)}$ is a strict $(\underline{\mathcal{A}}, P)$-supermartingale, as is permitted in a benchmark model. For an example we refer to Section 4.4.

## 3.4 Fair and Actuarial Pricing

In the following, we generalize the concepts of risk neutral and actuarial pricing. We introduce the natural notion of a *fair* price. Here the pricing formula (3.12) is, in principle, taken as the definition for the corresponding benchmarked fair price. Thus the fair price can be determined even if there is no equivalent risk neutral measure. We remark that other notions of fair prices were suggested previously, for instance, in Davis (1997), Karatzas & Shreve (1998) and Yan (1998). These typically assume the existence of an equivalent risk neutral pricing measure.

**Definition 3.4** *We call a portfolio process $S^{(\delta)}$* fair *if the corresponding benchmarked portfolio process $\hat{S}^{(\delta)}$ forms an $(\underline{\mathcal{A}}, P)$-martingale.*

Fair portfolios form an important subclass of portfolio processes. Their last observed benchmarked value is the best forecast of any of their future benchmarked values and is thus a fair price in a rather intuitive sense. The notion of a fair price is formulated via conditional expectations. It does not involve any utility function. However, one can say that an investor who accepts a fair price is indifferent towards investing in the GOP or buying the fairly priced corresponding security.

Definition 3.4 leads, by (3.5) and the martingale property of $\hat{S}^{(\delta)}$, to the following conclusion.

**Corollary 3.5** *A nonnegative, fair portfolio process* $S^{(\delta)} = \{S^{(\delta)}(t),\, t \in [0,T]\}$ *has zero arbitrage amount, that is*

$$A_\delta(t) = 0 \tag{3.13}$$

*for all* $t \in [0,T]$.

Let us now price payoffs that arise at a certain future date.

**Definition 3.6** *We define a* contingent claim $H_\tau$ *that matures at a stopping time* $\tau \in [0,T]$ *as an* $\mathcal{A}_\tau$*-measurable, nonnegative payoff with*

$$E\left(\frac{H_\tau}{S^{(\delta_*)}(\tau)}\,\bigg|\,\mathcal{A}_t\right) < \infty \tag{3.14}$$

*for* $t \in [0,\tau]$.

This allows us to state the following *benchmark pricing formula* for the fair domestic price of a given contingent claim.

**Corollary 3.7** *The fair price* $U_{H_\tau}(t)$ *at time* $t$ *for a given contingent claim* $H_\tau$*, when expressed in units of the domestic currency, is given by the* benchmark pricing formula

$$U_{H_\tau}(t) = S^{(\delta_*)}(t)\, E\left(\frac{H_\tau}{S^{(\delta_*)}(\tau)}\,\bigg|\,\mathcal{A}_t\right) \tag{3.15}$$

*for* $t \in [0,\tau]$.

The benchmark pricing formula (3.15) for a given contingent claim generalizes the risk neutral pricing formula (3.11). It involves a conditional expectation that is taken with respect to the real world probability measure $P$ and does not assume the existence of any equivalent pricing measure. However, if such an equivalent measure exists and a given price process is fair, then its price coincides with the corresponding risk neutral price given in (3.11).

In the case when the maturity date $\tau$ is fixed and the payoff equals one unit of the domestic currency, then we get by (3.15) the corresponding fair price of a *zero coupon bond*

$$P(t,\tau) = S^{(\delta_*)}(t)\, E\left(\frac{1}{S^{(\delta_*)}(\tau)}\,\bigg|\,\mathcal{A}_t\right) \tag{3.16}$$

at time $t \in [0,\tau]$. From (3.15) and (3.16) we obtain now directly the standard *actuarial pricing formula.*

**Corollary 3.8** *If a contingent claim $H_\tau$ matures on a fixed date $\tau \in [0,T]$ and is independent of the GOP value $S^{(\delta_*)}(\tau)$, then its fair price, in domestic currency, satisfies the* actuarial pricing formula

$$U_{H_\tau}(t) = P(t,\tau)\, E\left(H_\tau \,\big|\, \mathcal{A}_t\right) \tag{3.17}$$

*for $t \in [0,\tau]$.*

Note that the interest rate term structure can be stochastic in (3.17). Consequently, the benchmark model does not only integrate risk neutral pricing but also actuarial pricing. The concept of fair pricing permits the pricing of any contingent claim. This covers, for instance, credit risk, market risk, insurance, operational risk, weather risk, energy derivatives, real options, catastrophy risk and any other area of economic activity.

## 3.5 Fair Hedging

It is of great practical interest to identify a portfolio that permits the hedging of a given contingent claim.

**Definition 3.9** *For a given contingent claim $H_\tau$, a corresponding fair portfolio process $S^{(\delta_{H_\tau})}$ is called* replicating, *if at maturity date $\tau$,*

$$S^{(\delta_{H_\tau})}(\tau) = H_\tau \tag{3.18}$$

*a.s.*

From Definition 3.9, Definition 3.4, relations (3.5) and (3.13) and the martingale property of a benchmarked, fair, replicating portfolio one obtains the following result.

**Corollary 3.10** *For a given contingent claim $H_\tau$, a corresponding fair, replicating portfolio $S^{(\delta_{H_\tau})}$ must have the value*

$$S^{(\delta_{H_\tau})}(t) = U_{H_\tau}(t) \tag{3.19}$$

*for all times $t \in [0,\tau]$.*

The natural question that arises is, whether there exists a corresponding replicating portfolio for all contingent claims that are described in Definition 3.6.

**Definition 3.11** *A given benchmark model is called* complete *if for all square integrable contingent claims a corresponding replicating portfolio exists.*

The completeness of a market is closely linked to the existence of a *martingale representation* for each contingent claim. Such a martingale representation is, for instance, described in Karatzas & Shreve (1988) for functionals of Brownian motions, in Föllmer & Schweizer (1991) for a wide class of asset price models or in Jacod, Méléard & Protter (2000) for certain Markovian semimartingales. To be specific, we assume that we have for each contingent claim $H_\tau$ a martingale representation of the form

$$\frac{H_\tau}{S^{(\delta_*)}(\tau)} = \hat{U}_{H_\tau}(t) + \sum_{k=1}^{d} \int_t^\tau x^k_{H_\tau}(s-)\, dW^k(s) \tag{3.20}$$

with

$$\hat{U}_{H_\tau}(t) = \frac{U_{H_\tau}(t)}{S^{(\delta_*)}(t)} = E\left( \frac{H_\tau}{S^{(\delta_*)}(\tau)} \,\middle|\, \mathcal{A}_t \right) \tag{3.21}$$

for $t \in [0, \tau]$ and certain unique, progressively measurable process $x_{H_\tau} = \{x_{H_\tau}(t) = (x^1_{H_\tau}(t), \ldots, x^d_{H_\tau}(t))^\top,\ t \in [0, \tau]\}$ for which

$$E\left( \int_0^\tau \sum_{k=1}^{d} \left( x^k_{H_\tau}(s) \right)^2 ds \right) < \infty. \tag{3.22}$$

In a Markovian multi-factor setting, as will be discussed later, a martingale representation is easily obtained by application of the Itô formula. The following property of the generalized volatility matrix $b(t)$, see (2.7), is important for the implementation of a unique hedge.

**Definition 3.12** *The given benchmark model is called* invertible *if the generalized volatility matrix $b(t)$ is invertible, that is*

$$b^{-1}(t) < \infty \tag{3.23}$$

*a.s. for Lebesgue-almost-every $t \in [0, T]$.*

An invertible generalized volatility matrix uniquely links the sources of uncertainty to the primary security accounts. We prove the following result, which provides an explicit *fair hedging strategy* for each contingent claim.

**Theorem 3.13** *An invertible benchmark model is complete. For each square integrable contingent claim $H_\tau$ there exists a fair, replicating portfolio process $S^{(\delta_{H_\tau})}$ that has at time $t \in [0, \tau]$ the value*

$$S^{(\delta_{H_\tau})}(t) = U_{H_\tau}(t) \tag{3.24}$$

*with corresponding unique vector of proportions*

$$\pi_{\delta_{H_\tau}}(t) = \left( c_{H_\tau}(t)^\top b^{-1}(t) \right)^\top, \tag{3.25}$$

*where* $c_{H_\tau}(t) = (c^1_{H_\tau}(t), \ldots, c^d_{H_\tau}(t))^\top$ *has the components*

$$c^k_{H_\tau}(t) = \begin{cases} \frac{x^k_{H_\tau}(t)}{\hat{U}_{H_\tau}(t)} + \sigma^{0,k}(t) & \text{for} \quad k \in \{1, 2, \ldots, m\} \\ \left\{\frac{1}{\varphi^{0,k}_{t-}}\left(\frac{x^k_{H_\tau}(t-)}{\hat{U}_{H_\tau}(t-)} + 1\right) - 1\right\} & \text{for} \quad k \in \{m+1, \ldots, d\} \end{cases} \tag{3.26}$$

*for* $t \in [0, \tau]$.

**Proof**: For a given square integrable contingent claim $H_\tau$ we use the martingale representation (3.20) and formula (3.21) to obtain by application of the Itô formula for the savings account discounted fair price

$$\bar{U}_{H_\tau}(t) = \frac{U_{H_\tau}(t)}{S^{(0)}(t)} \tag{3.27}$$

the representation

$$\begin{aligned} \bar{U}_{H_\tau}(\tau) &= \bar{U}_{H_\tau}(t) + \sum_{k=1}^{m} \int_t^\tau \bar{U}_{H_\tau}(s) \left(\frac{x^k_{H_\tau}(s)}{\hat{U}_{H_\tau}(s)} + \sigma^{0,k}(s)\right) (\sigma^{0,k}(s)\, ds + dW^k(s)) \\ &\quad + \sum_{k=m+1}^{d} \int_t^\tau \bar{U}_{H_\tau}(s-) \left(\frac{x^k_{H_\tau}(s-)}{\hat{U}_{H_\tau}(s-)} + 1 - \varphi^{0,k}_{s-}\right) \\ &\quad \cdot \frac{1}{\varphi^{0,k}_{s-}} \left(dp^k(s) - \varphi^{0,k}_{s-}\, h^k_{s-}\, ds\right) \end{aligned} \tag{3.28}$$

for $t \in [0, \tau]$. On the other hand, it follows by the Itô formula, (2.2), (2.9), (2.7) and (2.3) that the discounted value

$$\bar{S}^{(\delta_{H_\tau})}(t) = \frac{S^{(\delta_{H_\tau})}(t)}{S^{(0)}(t)} \tag{3.29}$$

of a replicating portfolio must satisfy the relation

$$\begin{aligned} \bar{S}^{(\delta_{H_\tau})}(\tau) &= \bar{S}^{(\delta_{H_\tau})}(t) + \sum_{j=1}^{d} \int_t^\tau \delta_{H_\tau}(s-)\, d\bar{S}^{(j)}(s) \\ &= \bar{S}^{(\delta_{H_\tau})}(t) + \sum_{k=1}^{m} \int_t^\tau \bar{S}^{(\delta_{H_\tau})}(s) \sum_{j=1}^{d} \pi^j_{\delta_{H_\tau}}(s)\, b^{j,k}(s)\, (\sigma^{0,k}(s)\, ds + dW^k(s)) \\ &\quad + \sum_{k=m+1}^{d} \int_t^\tau \bar{S}^{(\delta_{H_\tau})}(s-) \sum_{j=1}^{d} \pi^j_{\delta_{H_\tau}}(s-) \left(\varphi^{j,k}_{s-} - \varphi^{0,k}_{s-}\right) \\ &\quad \cdot \frac{1}{\varphi^{0,k}_{s-}} \left(dp^k(s) - \varphi^{0,k}_{s-}\, h^k_{s-}\, ds\right) \end{aligned} \tag{3.30}$$

for $t \in [0, \tau]$, where $d\bar{S}^{(0)}(s) = 0$. By comparison of equations (3.28) and (3.30) and using (2.7) it follows that

$$c_{H_\tau}(t) = \left(\pi^\top_{\delta_{H_\tau}}(t)\, b(t)\right)^\top \tag{3.31}$$

for $t \in [0, \tau]$. This proves (3.25) and thus with (3.19) and (3.23) Theorem 3.13. $\square$

A martingale representation ensures in a complete benchmark model the existence of a corresponding fair, replicating hedge portfolio. An incomplete benchmark model can be obtained from the above setting by excluding certain primary security accounts from trading. One can then only replicate the, so called, *hedgable part* of the contingent claim. By employing the fair price when setting up the hedge, one uses the best forecast that is available to offset in average the benchmarked *unhedgable part*. Detailed results on incomplete benchmark models will be presented in a forthcoming paper.

# 4 Multi-Factor Benchmark Model

## 4.1 Factors

To construct computationally tractable models, we consider in the following a class of *multi-factor benchmark models*. The factors $Z^0, \dots, Z^n$ can represent any reasonable financial quantities in the market. They are assumed to form an adapted, right-continuous vector Markov process that is characterized by the system of SDEs

$$\begin{aligned} dZ^\ell(t) &= \alpha^\ell(t, Z^0(t), \dots, Z^n(t))\, dt \\ &\quad + \sum_{k=1}^{m} \beta^{\ell,k}(t, Z^0(t), \dots, Z^n(t))\, dW^k(t) \\ &\quad + \sum_{k=m+1}^{d} \gamma^{\ell,k}(t-, Z^0(t-), \dots, Z^n(t-))\, dp^k(t) \end{aligned} \tag{4.1}$$

for $t \in [0, T]$ with given initial values $Z^\ell(0) \in \Re$ for $\ell \in \{0, 1, \dots, n\}$. The $k$th intensity $h^k_t = h^k(t, Z^0(t), \dots, Z^n(t)) \in (0, \infty)$ is assumed to be given as a function of time and factors, $k \in \{m+1, \dots, d\}$. Note that all factors can be driven by all Wiener and counting processes. We assume that the functions $\alpha^\ell$, $\beta^{\ell,k}$, $\gamma^{\ell,r}$ and $h^k$ are such that a pathwise, unique solution of the SDE (4.1) for the factor process $Z = \{Z(t) = (Z^0(t), \dots, Z^n(t))^\top, t \in [0, T]\}$ exists.

## 4.2 Volatilities and Jump Ratios

In a multi-factor benchmark model let the $j$th benchmarked primary security account $\hat{S}^{(j)}$, see (2.21), for each $j \in \{0, 1, \dots, d\}$ be characterized by a function $\hat{S}^{(j)} : [0, T] \times \Re^{n+1} \to [0, \infty)$, which is differentiable with respect to the time $t$ and twice differentiable with respect to the factors $Z^0, \dots, Z^n$, such that the SDE (2.21) is satisfied. To express conveniently volatilities and jump ratios, we introduce for an appropriately differentiable function $f : [0, T] \times \Re^{n+1} \to \Re$ the operator

$$\begin{aligned} L^0 f(t, Z^0, \dots, Z^n) & \\ = \; & \frac{\partial f(t, Z^0, \dots, Z^n)}{\partial t} + \sum_{\ell=0}^{n} \alpha^\ell(t, Z^0, \dots, Z^n) \frac{\partial f(t, Z^0, \dots, Z^n)}{\partial Z^\ell} \\ & + \frac{1}{2} \sum_{\ell, r=0}^{n} \sum_{k=1}^{m} \beta^{\ell,k}(t, Z^0, \dots, Z^n)\, \beta^{r,k}(t, Z^0, \dots, Z^n) \frac{\partial^2 f(t, Z^0, \dots, Z^n)}{\partial Z^\ell \, \partial Z^r} \\ & + \sum_{k=m+1}^{d} \Delta_f^k(t, Z^0, \dots, Z^n)\, h^k(t, Z^0, \dots, Z^n) \end{aligned} \tag{4.2}$$

with $k$th jump size

$$\begin{aligned} \Delta_f^k(t, Z^0, \dots, Z^n) & \\ = \; & f(t, Z^0 + \gamma^{0,k}(t, Z^0, \dots, Z^n), \dots, Z^n + \gamma^{n,k}(t, Z^0, \dots, Z^n)) \\ & - f(t, Z^0, \dots, Z^n) \end{aligned} \tag{4.3}$$

for $k \in \{m + 1, \dots, d\}$, and operator

$$L^k f(t, Z^0, \dots, Z^n) = \sum_{\ell=0}^{n} \beta^{\ell,k}(t, Z^0, \dots, Z^n) \frac{\partial f(t, Z^0, \dots, Z^n)}{\partial Z^\ell} \tag{4.4}$$

for $k \in \{1, 2, \dots, m\}$ and $(t, Z^0, \dots, Z^n) \in (0, T) \times \Re^{n+1}$. To ensure (2.21) we need to assume that

$$L^0 \hat{S}^{(j)}(t, Z^0, \dots, Z^n) = 0 \tag{4.5}$$

for $(t, Z^0, \dots, Z^n) \in (0, T) \times \Re^{n+1}$. Then we obtain for the $j$th benchmarked primary security account by the Itô formula and (2.1) the representation

$$\begin{aligned} \hat{S}^{(j)}(t) \; = \; & \hat{S}^{(j)}(t, Z^0(t), \dots, Z^n(t)) \\ = \; & \hat{S}^{(j)}(0, Z^0(0), \dots, Z^n(0)) \\ & + \sum_{k=1}^{m} \int_0^t L^k \hat{S}^{(j)}(s, Z^0(s), \dots, Z^n(s))\, dW^k(s) \\ & + \sum_{k=m+1}^{d} \int_0^t \Delta_{\hat{S}^{(j)}}^k(s-, Z^0(s-), \dots, Z^n(s-)) \left(dp^k(s) - h_{s-}^k\, ds\right) \end{aligned} \tag{4.6}$$

for $t \in [0, T]$ and $j \in \{0, 1, \dots, d\}$. By comparing (4.6) with (2.21) one obtains the $j, k$th volatility

$$\sigma^{j,k}(t) = \sum_{\ell=0}^{n} \beta^{\ell,k}(t, Z^0(t), \dots, Z^n(t)) \frac{\partial \log(\hat{S}^{(j)}(t, Z^0(t), \dots, Z^n(t)))}{\partial Z^\ell}, \tag{4.7}$$

for $k \in \{1, 2, \dots, m\}$ and the $j, k$th jump ratio

$$\varphi_{t-}^{j,k} = \frac{\Delta^k_{\hat{S}^{(j)}}(t-, Z^0(t-), \dots, Z^n(t-))}{\hat{S}^{(j)}(t-, Z^0(t-), \dots, Z^n(t-))} + 1 \tag{4.8}$$

as long as $\hat{S}^{(j)}(t) > 0$, for $k \in \{m+1, \dots, d\}$, $j \in \{0, 1, \dots, d\}$ and $t \in [0, T]$. Furthermore, the short term interest rate $r(t) = r(t, Z^0(t), \dots, Z^n(t))$ is assumed to be given by a function $r : [0, T] \times \Re^{n+1} \to [0, \infty)$ of time and factors. Under the minor technical condition (2.6) we obtain a benchmark model with intensity based jumps.

## 4.3 Pricing and Hedging

Let us consider a nonnegative contingent claim $H_\tau = H(\tau, Z^0(\tau), \dots, Z^n(\tau))$, which is a function of the maturity $\tau$ and the factors $Z^0(\tau), \dots, Z^n(\tau)$. By Corollary 3.7 we obtain the following result.

**Corollary 4.1** *The benchmarked fair price $\hat{U}_{H_\tau}(t)$ at time $t \in [0, \tau]$ for the above nonnegative contingent claim $H_\tau$ is given by the conditional expectation*

$$\hat{U}_{H_\tau}(t) = \hat{U}_{H_\tau}(t, Z^0(t), \dots, Z^n(t)) = E\left( \frac{H(\tau, Z^0(\tau), \dots, Z^n(\tau))}{S^{(\delta_*)}(\tau)} \,\middle|\, \mathcal{A}_t \right). \tag{4.9}$$

If we assume for a given contingent claim $H_\tau$ that the function $\hat{U}_{H_\tau} : (0, T) \times \Re^{n+1} \to \Re$ is differentiable with respect to time and twice differentiable with respect to the factors, then $\hat{U}_{H_\tau}(\cdot, \cdot)$ satisfies the integro-partial differential equation

$$L^0 \, \hat{U}_{H_\tau}(t, Z^0(t), \dots, Z^n(t)) = 0 \tag{4.10}$$

for $(t, Z^0(t), \dots, Z^n(t))$ in the interior of a continuation region $\Gamma$ and appropriate boundary conditions for $(t, Z^0(t), \dots, Z^n(t))$ outside of $\Gamma$. Furthermore, we obtain by the Itô formula and from (3.20) and (3.26) that

$$c^k_{H_\tau}(t) = \begin{cases} \frac{L^k \hat{U}_{H_\tau}(t)}{\hat{U}_{H_\tau}(t)} + \sigma^{0,k}(t) & \text{for} \quad k \in \{1, 2, \dots, m\} \\ \frac{1}{\varphi^{0,k}_{t-}} \left( \frac{\Delta^k_{\hat{U}_{H_\tau}}(t-, Z^0(t-), \dots, Z^n(t-))}{\hat{U}_{H_\tau}(t-)} + 1 \right) - 1 & \text{for} \quad k \in \{m+1, \dots, d\} \end{cases} \tag{4.11}$$

for $t \in [0, \tau)$. If we assume that the benchmark model is invertible, see Definition 3.12, then we obtain by Theorem 3.13 the vector of proportions

$$\pi_{\delta_H}(t) = \left(c_{H_\tau}(t)^\top b^{-1}(t)\right)^\top \tag{4.12}$$

for $t \in [0, \tau)$ for the fair, replicating hedge portfolio $S^{\delta_{H_\tau}}$. For all square integrable contingent claims there exists in an invertible benchmark model a perfectly replicating fair hedge portfolio. Thus we can state the following result.

**Corollary 4.2** *A given invertible, multi-factor benchmark model forms a complete market.*

A rich set of market dynamics can be modeled by using the above multi-factor benchmark model. It is characterized by the functions $\hat{S}^{(j)}$, $\alpha^j$, $\beta^{\ell,k}$, $\gamma^{\ell,r}$ and $h^k$ together with corresponding initial values.

## 4.4 A Defaultable Minimal Market Model

To illustrate the benchmark approach let us consider a simple defaultable extended version of the *Minimal Market Model* (MMM), proposed in Platen (2001).

We use as factor $Z^0$ a *square root process* of dimension four, which models the GOP by the expression

$$S^{(\delta_*)}(t) = Z^0(t)\, \exp(\eta^2\, t) \tag{4.13}$$

and satisfies the SDE

$$dZ^0(t) = \eta^2\,(1 - Z^0(t))\, dt + \sqrt{Z^0(t)}\, \eta\, dW^1(t) \tag{4.14}$$

for $t \in [0, T]$ with initial value $Z^0(0) = 1$. Here $W^1$ is a standard $(\underline{\mathcal{A}}, P)$-Wiener process. The *scaling parameter* $\eta$ is a given constant and plays a similar role as the volatility in the Black-Scholes model. The above factor process is uniquely determined, remains a.s. strictly positive and does not explode, see Protter (1990). Since

$$dS^{(\delta_*)}(t) = S^{(\delta_*)}(t) \left( \frac{\eta^2}{Z^0(t)}\, dt + \frac{\eta}{\sqrt{Z^0(t)}}\, dW^1(t) \right) \tag{4.15}$$

it follows by (2.15) zero interest rate $r(t) = 0$ with constant savings account and thus the benchmarked savings account $\hat{S}^{(0)}(t) = \frac{\exp(-\eta^2\, t)}{Z^0(t)}$ for $t \in [0, T]$. As primary security account $S^{(1)}$ we choose the GOP itself, that is, $\hat{S}^{(1)}(t) = 1$ for all $t \in [0, T]$.

The primary security account $S^{(2)}$ is assumed to pay at time $T$ one unit of the domestic currency if a certain type of event does not occur. Such an event may be, for instance, the default of a company, a merger, an earthquake or an accident. The occurrence of the event is counted by the Poisson process $p = \{p(t),\, t \in [0,T]\}$, which has constant intensity $h > 0$. The price $S^{(2)}(t)$ is defined as the fair defaultable zero coupon bond price at time $t$ with maturity $T$, which defaults if an event of the given type occurs. To model the value of the primary security account value $S^{(2)}(T)$ at time $T$ we introduce the factor $Z^1(t)$, which satisfies the pure jump SDE

$$dZ^1(t) = -Z^1(t-)\, dp(t) \tag{4.16}$$

for $t \in [0,T]$ with initial value $Z^1(0) = 1$. At the first jump time of the Poisson process $p$ the factor $Z^1$ reduces its value from 1 to 0 and remains there afterwards. The benchmarked primary security account process $\hat{S}^{(2)}$ is then specified as the fair price process of the contingent claim

$$H_T = S^{(2)}(T) = Z^1(T).$$

Using the transition density of $Z^0$ and the Poisson probability that no event is occurring, we get by (3.15)

$$\begin{aligned} \hat{S}^{(2)}(t) &= E\left( \frac{Z^1(T)}{S^{(\delta_*)}(T)} \,\middle|\, \mathcal{A}_t \right) \\ &= \frac{\exp(-h\,(T-t))}{Z^0(t)\, \exp(\eta^2\, t)} \left( 1 - \exp\left( - \frac{2\, Z^0(t)\, \exp(\eta^2\, t)}{\eta^2\,(T-t)} \right) \right) \end{aligned} \tag{4.17}$$

for $t \in [0,T]$. The volatility $\sigma^{0,1}(t)$ is here stochastic and has by (4.7) the form

$$\sigma^{0,1}(t) = -\eta \left( Z^0(t) \right)^{-\frac{1}{2}} \tag{4.18}$$

for $t \in [0,T]$. Furthermore, $\sigma^{2,1}(t)$ is determined by (4.7) and (4.17), and we have $\sigma^{1,2}(t) = 0$, $\varphi_t^{0,2} = \varphi_t^{1,2} = 1$ and $\varphi_t^{2,2} = 0$ for $t \in [0,T]$.

It follows from Revuz & Yor (1999) that $\hat{S}^{(0)}(t)$ is the inverse of a time transformed, squared Bessel process of dimension four, which forms a strict local martingale. Obviously, $\hat{S}^{(1)}$ and $\hat{S}^{(2)}$ are martingales. Consequently, we have a benchmark model with intensity based jumps, see Definition 2.2. The Radon-Nikodym derivative process $\Lambda = \{\Lambda(t) = \hat{S}^{(0)}(t),\, t \in [0,T\}$, see (3.8), is a strict $(\underline{\mathcal{A}}, P)$-local martingale and not a martingale. Under the risk neutral measure $\tilde{P}$ the discounted GOP $\frac{S^{(\delta_*)}(t)}{S^{(0)}(t)} = S^{(\delta_*)}(t)$ is a time transformed, squared Bessel process of dimension zero, which is absorbed at zero with strictly positive $\tilde{P}$-probability. On the other hand, $S^{(\delta_*)}(t)$ never reaches zero under the real world measure $P$. Thus the measures $P$ and $\tilde{P}$ are not equivalent. Consequently, an equivalent risk neutral measure does not exist in the given benchmark model and the standard risk neutral approach is not applicable. Despite this, the above benchmark approach provides both a consistent pricing system and adequate hedging prescriptions.

The primary security account $S^{(2)}$ is a fair defaultable zero coupon bond. For comparison, we can also compute the fair price of the corresponding non-defaultable zero coupon bond at time $t$, which, according to (3.16) and (4.17), equals

$$\begin{aligned} P(t,T) &= S^{(\delta_*)}(t)\, E\left(\frac{1}{S^{(\delta_*)}(T)}\,\bigg|\,\mathcal{A}_t\right) \\ &= \left(1-\exp\left(-\frac{2\,Z^0(t)\,\exp(\eta^2\,t)}{\eta^2\,(T-t)}\right)\right) \geq S^{(2)}(t) \qquad (4.19) \end{aligned}$$

for all $t \in [0,T]$. The resulting spread between the corresponding defaultable and non-defaultable forward rate is given by the intensity $h$ of the Poisson process $P$. Finally, we remark that the value of the fair, non-defaultable bond $P(t,T)$ is smaller than the constant value of the savings account, which also delivers one unit of the currency at time $T$. Consequently, by (3.5) and (4.19), the nonzero arbitrage amount of the savings account at time $t$ equals

$$A_\delta(t) = \exp\left(-\frac{2\,Z^0(t)\,\exp(\eta^2\,t)}{\eta^2\,(T-t)}\right) > 0$$

for $t \in [0,T)$, where $\delta(t) = (1,0,0)^\top$.

## Conclusion

The proposed benchmark model with intensity based jumps uses the growth optimal portfolio as the basic building block. An integrated framework is provided for the consistent modeling of various forms of risk, including event driven risk in finance and insurance. Furthermore, the existence of an equivalent risk neutral measure is not required. Benchmarked fair prices are defined as martingales, which coincide with corresponding benchmarked risk neutral prices in the case when an equivalent risk neutral measure exists. The actuarial pricing formula can be recovered for a wide range of insurance models. The proposed multi-factor benchmark model with intensity based jumps offers a rich, integrated modeling framework. The key quantitative tasks of calibration, estimation, derivative pricing, hedging, portfolio optimization and risk measurement can be performed under the real world probability measure.

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# References


Ansel, J. P. & C. Stricker (1994). Couverture des actifs contingents. *Ann. Inst. H. Poincaré Probab. Statist.* **30**, 303–315.

Bajeux-Besnainou, I. & R. Portait (1997). The numeraire portfolio: A new perspective on financial theory. *The European Journal of Finance* **3**, 291–309.

Becherer, D. (2001). The numeraire portfolio for unbounded semimartingales. *Finance and Stochastics* **5**, 327–341.

Bühlmann, H. (1992). Stochastic discounting. *Insurance: Mathematics and Economics* **11**, 113–127.

Bühlmann, H., F. Delbaen, P. Embrechts, & A. Shiryaev (1998). On Esscher transforms in discrete finance models. *ASTIN Bulletin* **28**(2), 171–186.

Bühlmann, H. & E. Platen (2002). A discrete time benchmark approach for finance and insurance. Technical report, University of Technology, Sydney. QFRG Research Paper 74.

Davis, M. H. A. (1997). Option pricing in incomplete markets. In M. A. H. Dempster and S. R. Pliska (Eds.), *Mathematics of derivative securities*, pp. 227–254. Cambridge University Press.

Delbaen, F. & W. Schachermayer (1994). A general version of the fundamental theorem of asset pricing. *Math. Ann* **300**, 463–520.

Delbaen, F. & W. Schachermayer (1998). The fundamental theorem of asset pricing for unbounded stochastic processes. *Math. Ann* **312**, 215–250.

Duffie, D. (1996). *Dynamic Asset Pricing Theory* (2nd ed.). Princeton, University Press.

Föllmer, H. & M. Schweizer (1991). Hedging of contingent claims under incomplete information. In M. Davis and R. Elliott (Eds.), *Applied Stochastic Analysis*, Volume 5 of *Stochastics Monogr.*, pp. 389–414. Gordon and Breach, London/New York.

Föllmer, H. & D. Sondermann (1986). Hedging of non-redundant contingent claims. In W. Hildebrandt and A. Mas-Colell (Eds.), *Contributions to Mathematical Economics*, pp. 205–223. North Holland.

Geman, S., N. El Karoui, & J. C. Rochet (1995). Changes of numeraire, changes of probability measures and pricing of options. *J. Appl. Probab.* **32**, 443–458.

Gerber, H. U. (1990). *Life Insurance Mathematics.* Springer, Berlin.

Goll, T. & J. Kallsen (2002). A complete explicit solution to the log-optimal portfolio problem. (working paper), Universität Freiburg i. Br.

Harrison, J. M. & D. M. Kreps (1979). Martingale and arbitrage in multiperiod securities markets. *J. Economic Theory* **20**, 381–408.

Harrison, J. M. & S. R. Pliska (1981). Martingales and stochastic integrals in the theory of continuous trading. *Stochastic Process. Appl.* **11**, 215–260.

Jacod, J., S. Méléard, & P. Protter (2000). Explicit form and robustness of martingale representations. *Ann. Probab.* **28**(4), 1747–1780.

Karatzas, I. & S. E. Shreve (1988). *Brownian Motion and Stochastic Calculus.* Springer.

Karatzas, I. & S. E. Shreve (1998). *Methods of Mathematical Finance*, Volume 39 of *Appl. Math.* Springer.

Kramkov, D. O. & W. Schachermayer (1999). The asymptotic elasticity of utility functions and optimal investment in incomplete markets. *Ann. Appl. Probab.* **9**, 904–950.

Long, J. B. (1990). The numeraire portfolio. *J. Financial Economics* **26**, 29–69.

Merton, R. C. (1973). An intertemporal capital asset pricing model. *Econometrica* **41**, 867–888.

Platen, E. (2001). A minimal financial market model. In *Trends in Mathematics*, pp. 293–301. Birkhäuser.

Platen, E. (2002). A benchmark framework for integrated risk management. University of Technology Sydney, (working paper).

Protter, P. (1990). *Stochastic Integration and Differential Equations.* Springer.

Revuz, D. & M. Yor (1999). *Continuous Martingales and Brownian Motion* (3rd ed.). Springer.

Rogers, L. C. G. (1997). The potential approach to the term structure of interest rates and their exchange rates. *Math. Finance* **7**, 157–176.

Ross, S. A. (1976). The arbitrage theory of capital asset pricing. *J. Economic Theory* **13**, 341–360.

Shiryaev, A. N. & A. S. Cherny (2001). Vector stochastic integrals and the fundamental theorems of asset pricing. Steklov Mathematical Institute Moscow, (working paper).

Yan, J. A. (1998). A new look at the fundamental theorem of asset pricing. *J. Korean Math. Soc.* **35**(3), 659–673.